Constructing p-adic L-functions

Monday, March 24, 2008
4:32 PM

Relate p-adic L-functions to characteristic function of Selmer gps.

First step: constructing p-adic L-functions

conj's: Coates, Perrin-Riou, Panchishkin.

Automorphic Forms on Unitary Gps

Special
(there are other kinds of gps)

general
- connect to GL_n
- these automorphic forms are generic as Hecke characters of CM fields are generic.

$\mathcal{K}$ quad. imag. fld

$c \in Gal(\mathcal{K}/\mathbb{Q})$ $\qquad$ $W/\mathcal{K}$ Hermitian space of dim n

$\langle u,v \rangle$ linear in first variable, c-linear in second variable.

hermitian symm.

$-W$ space W, $(\ , \ )$ is $-\langle \ , \ \rangle$.

$2W = W \oplus (-W)$ simplest kind of hermitian space.
$\| $
$W \oplus W^{d}$ $\quad ((w,w), (v,v)) = 0 \quad \forall \; v,w \in W$.

totally isotropic subspaces $\qquad H = U(2W) \qquad G = U(W) \qquad P = Stab_H(W^d)$
max. parabolic subgp.

$m(A) = \begin{pmatrix} A & \\ & {}^{c}{}^{t}A^{-1} \end{pmatrix} \in P \qquad M = \{m(A)\}$ $\qquad P \ni p = \begin{pmatrix} A & X \\ 0 & c({}^{t}A^{-1}) \end{pmatrix}$

R a $\mathbb{Q}$ algebra $\qquad U(R) = \left\{ \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \in P \right\}$ X runs over $Herm_n(R \otimes_{\mathbb{Q}} \mathcal{K})$. $\qquad A \in GL_n(\mathcal{K})$.

$A = A(p)$

$w = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

I. Construction of p-adic L-fcns

II. p-adic L-fcn of (nearly ordinary Hida families)

doubling formula for L-functions of unitary gps

generalization of Katz's construction in the case $n=1$.

Consider the subgps of H with values in $\mathbb{Q}$, completion of $\mathbb{Q}$, $\mathbb{A}$.

Let v be a completion of $\mathbb{Q}$,

$$\delta_v(p) = |N_{K/\mathbb{Q}} \circ \det A(p)|^{\frac{n}{2}} \quad (\text{modulus function})$$

$\delta_{\mathbb{A}}$ adelic version $\quad \prod \delta_v$

$H(\mathbb{A}) = P(\mathbb{A}) \cdot K_{\mathbb{A}}$

$\chi : K_{\mathbb{A}}^\times / K^\times \to \mathbb{C}^\times$ Hecke character.

$K_{\mathbb{A}}$ = adelic max. compact.

χ a chara. of $M_{\mathbb{A}}$ by composition with det of $GL_n(\mathbb{A})$.

$h = p(h) \cdot \gamma(h)$ ~~not~~ well defined.

$$\delta_{\mathbb{A}}(p, \chi, s) = \delta_{\mathbb{A}}(p) \cdot \chi \circ \det(A(p)) \, |N_{K/\mathbb{Q}} \circ \det(A(p))|_{\mathbb{A}}^{s}$$

$$\sum_{\gamma \in P(\mathbb{Q}) \backslash H(\mathbb{Q})} \delta_{\mathbb{A}}(p(\gamma h), \chi, s) = E(h, \chi, s) \qquad K_{\mathbb{A}} \qquad \text{Eisenstein series}$$

General Eisenstein series attached to functions (non-normalized induction)

$$f(h, \chi, s) \in \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \delta(\cdot, \chi, s) = \{ f : H(\mathbb{A}) \to \mathbb{C} \mid f(ph) = \delta(p, \chi, s) f(h) \}.$$

$$\sum \frac{y^s}{|cz+d|^{2s}} \qquad E_f(h, \chi, s) = \sum_{P(\mathbb{Q}) \backslash H(\mathbb{Q})} f(\gamma h, \chi, s) \quad \text{converges for } \mathrm{Re}\, s \gg 0.$$

For appropriate choices of f_∞ and s

$n = 1$, $H = U(1,1) \supset SL_2$

$E_f|_{SL_2(\mathbb{R})}$ is essentially

appropriately $s \leftrightarrow k$

$$y^{-\frac{k}{2}} \sum \frac{1}{(cz+d)^{k}}$$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ldots$